ML-2
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Perturbation Methods

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Introduction

- Remember that we want to solve a functional equation of the form:

$$\mathcal{H}(d) = \mathbf{0}$$

for an unknown decision rule d.

- Perturbation solves the problem by specifying:

$$d^n(x, \theta) = \sum_{i=0}^{n} \theta_i (x - x_0)^i$$

- We use implicit-function theorems to find coefficients θ_i's.

- Inherently local approximation. However, often good global properties.

Motivation

- Many complicated mathematical problems have:

  1. either a particular case
  2. or a related problem.

  that is easy to solve.

- Often, we can use the solution of the simpler problem as a building block of the general solution.

- Very successful in physics.

- Sometimes perturbation is known as asymptotic methods.

The World Simplest Perturbation

- What is $\sqrt{26}$?
- Without your Iphone calculator, it is a boring arithmetic calculation.
- But note that:

$$\sqrt{26} = \sqrt{25\left(1 + 0.04\right)} = 5 * \sqrt{1.04} \approx 5 * 1.02 = 5.1$$

- Exact solution is 5.099.
- We have solved a much simpler problem ($\sqrt{25}$) and added a small coefficient to it.
- More in general

$$\sqrt{y} = \sqrt{x^2\left(1 + \varepsilon\right)} = x\sqrt{1 + \varepsilon}$$

where x is an integer and ε the perturbation parameter.

Applications to Economics

- Judd and Guu (1993) showed how to apply it to economic problems.
- Recently, perturbation methods have been gaining much popularity.
- In particular, second- and third-order approximations are easy to compute and notably improve accuracy.
- A first-order perturbation theory and linearization deliver the same output.
- Hence, we can use much of what we already know about linearization.

Regular versus Singular Perturbations

- Regular perturbation: a *small* change in the problem induces a *small* change in the solution.

- Singular perturbation: a *small* change in the problem induces a *large* change in the solution.

- Example: excess demand function.

- Most problems in economics involve regular perturbations.

- Sometimes, however, we can have singularities. Example: introducing a new asset in an incomplete markets model.

References

- General:
  1. *A First Look at Perturbation Theory* by James G. Simmonds and James E. Mann Jr.
  2. *Advanced Mathematical Methods for Scientists and Engineers: Asymptotic Methods and Perturbation Theory* by Carl M. Bender, Steven A. Orszag.
- Economics:
  1. Perturbation Methods for General Dynamic Stochastic Models" by Hehui Jin and Kenneth Judd.
  2. Perturbation Methods with Nonlinear Changes of Variables" by Kenneth Judd.
  3. A gentle introduction: "Solving Dynamic General Equilibrium Models Using a Second-Order Approximation to the Policy Function" by Martín Uribe and Stephanie Schmitt-Grohe.

A Baby Example: A Basic RBC

Model:

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \log c_t$$

$$\text{s.t. } c_t + k_{t+1} = e^{z_t} k_t^{\alpha} + (1-\delta) k_t, \ \forall \ t > 0$$
$$z_t = \rho z_{t-1} + \sigma \varepsilon_t, \ \varepsilon_t \sim \mathcal{N}(0,1)$$

Equilibrium conditions:

$$\frac{1}{c_t} = \beta \mathbb{E}_t \frac{1}{c_{t+1}} \left(1 + \alpha e^{z_{t+1}} k_{t+1}^{\alpha-1} - \delta\right)$$
$$c_t + k_{t+1} = e^{z_t} k_t^{\alpha} + (1-\delta) k_t$$
$$z_t = \rho z_{t-1} + \sigma \varepsilon_t$$

Computing a Solution

- The previous problem does not have a known "paper and pencil" solution except when (unrealistically) $\delta = 1$.
- Then, income and substitution effect from a technology shock cancel each other (labor constant and consumption is a fixed fraction of income).
- Equilibrium conditions with $\delta = 1$:

$$\frac{1}{c_t} = \beta \mathbb{E}_t \frac{\alpha e^{z_{t+1}} k_{t+1}^{\alpha-1}}{c_{t+1}}$$

$$c_t + k_{t+1} = e^{z_t} k_t^{\alpha}$$

$$z_t = \rho z_{t-1} + \sigma \varepsilon_t$$

- By "guess and verify":

$$c_t = (1 - \alpha\beta) e^{z_t} k_t^{\alpha}$$

$$k_{t+1} = \alpha\beta e^{z_t} k_t^{\alpha}$$

Another Way to Solve the Problem

- Now let us suppose that you missed the lecture when "guess and verify" was explained.

- You need to compute the RBC.

- What you are searching for? A decision rule for consumption:

$$c_t = c\left(k_t, z_t\right)$$

and another one for capital:

$$k_{t+1} = k\left(k_t, z_t\right)$$

Note that our d is just the stack of $c\left(k_t, z_t\right)$ and $k\left(k_t, z_t\right)$.

Equilibrium Conditions

- We substitute in the equilibrium conditions the budget constraint and the law of motion for technology.
- And we write the decision rules explicitly as function of the states.
- Then:

$$\frac{1}{c(k_t, z_t)} = \beta \mathbb{E}_t \frac{\alpha e^{\rho z_t + \sigma \varepsilon_{t+1}} k(k_t, z_t)^{\alpha - 1}}{c(k(k_t, z_t), \rho z_t + \sigma \varepsilon_{t+1})}$$

$$c(k_t, z_t) + k(k_t, z_t) = e^{z_t} k_t^{\alpha}$$

- System of functional equations.

Main Idea

- Transform the problem rewriting it in terms of a small perturbation parameter.
- Solve the new problem for a particular choice of the perturbation parameter.
- This step is usually ambiguous since there are different ways to do so.
- Use the previous solution to approximate the solution of original the problem.

A Perturbation Approach

- Hence, we want to transform the problem.
- Which perturbation parameter? Standard deviation σ.
- Why σ? Discrete versus continuous time.
- Set $\sigma = 0 \Rightarrow$deterministic model, $z_t = 0$ and $e^{z_t} = 1$.
- We know how to solve the deterministic steady state.

A Parametrized Decision Rule

- We search for decision rule:

$$c_t = c\left(k_t, z_t; \sigma\right)$$

and

$$k_{t+1} = k\left(k_t, z_t; \sigma\right)$$

- Note new parameter σ.

- We are building a local approximation around $\sigma = 0$.

Taylor's Theorem

- Equilibrium conditions:

$$\mathbb{E}_t \left(\frac{1}{c\left(k_t, z_t; \sigma\right)} - \beta \frac{\alpha e^{\rho z_t + \sigma \varepsilon_{t+1}} k\left(k_t, z_t; \sigma\right)^{\alpha - 1}}{c\left(k\left(k_t, z_t; \sigma\right), \rho z_t + \sigma \varepsilon_{t+1}; \sigma\right)} \right) = 0$$

$$c\left(k_t, z_t; \sigma\right) + k\left(k_t, z_t; \sigma\right) - e^{z_t} k_t^{\alpha} = 0$$

- We will take derivatives with respect to k_t, z_t, and σ.

- Apply Taylor's theorem to build solution around deterministic steady state. How?

Asymptotic Expansion I

$$\begin{aligned} c_t &= c\left(k_t, z_t; \sigma\right)\big|_{k,0,0} = c\left(k,0;0\right) \\ &+ c_k\left(k,0;0\right)\left(k_t - k\right) + c_z\left(k,0;0\right) z_t + c_\sigma\left(k,0;0\right)\sigma \\ &+ \frac{1}{2} c_{kk}\left(k,0;0\right)\left(k_t - k\right)^2 + \frac{1}{2} c_{kz}\left(k,0;0\right)\left(k_t - k\right) z_t \\ &+ \frac{1}{2} c_{k\sigma}\left(k,0;0\right)\left(k_t - k\right)\sigma + \frac{1}{2} c_{zk}\left(k,0;0\right) z_t \left(k_t - k\right) \\ &+ \frac{1}{2} c_{zz}\left(k,0;0\right) z_t^2 + \frac{1}{2} c_{z\sigma}\left(k,0;0\right) z_t \sigma \\ &+ \frac{1}{2} c_{\sigma k}\left(k,0;0\right)\sigma\left(k_t - k\right) + \frac{1}{2} c_{\sigma z}\left(k,0;0\right)\sigma z_t \\ &+ \frac{1}{2} c_{\sigma^2}\left(k,0;0\right)\sigma^2 + \ldots \end{aligned}$$

Asymptotic Expansion II

$$
\begin{aligned}
k_{t+1} \quad = \quad & k\left(k_t, z_t; \sigma\right)\big|_{k,0,0} = k\left(k, 0; 0\right) \\
& + k_k\left(k, 0; 0\right)\left(k_t - k\right) + k_z\left(k, 0; 0\right) z_t + k_\sigma\left(k, 0; 0\right)\sigma \\
& + \frac{1}{2} k_{kk}\left(k, 0; 0\right)\left(k_t - k\right)^2 + \frac{1}{2} k_{kz}\left(k, 0; 0\right)\left(k_t - k\right) z_t \\
& + \frac{1}{2} k_{k\sigma}\left(k, 0; 0\right)\left(k_t - k\right)\sigma + \frac{1}{2} k_{zk}\left(k, 0; 0\right) z_t\left(k_t - k\right) \\
& + \frac{1}{2} k_{zz}\left(k, 0; 0\right) z_t^2 + \frac{1}{2} k_{z\sigma}\left(k, 0; 0\right) z_t \sigma \\
& + \frac{1}{2} k_{\sigma k}\left(k, 0; 0\right)\sigma\left(k_t - k\right) + \frac{1}{2} k_{\sigma z}\left(k, 0; 0\right)\sigma z_t \\
& + \frac{1}{2} k_{\sigma^2}\left(k, 0; 0\right)\sigma^2 + \ldots
\end{aligned}
$$

Comment on Notation

- From now on, to save on notation, I will write

$$F(k_t, z_t; \sigma) = \mathbb{E}_t \left[\begin{array}{c} \frac{1}{c(k_t,z_t;\sigma)} - \beta \frac{\alpha e^{\rho z_t + \sigma \varepsilon_{t+1}} k(k_t,z_t;\sigma)^{\alpha-1}}{c(k(k_t,z_t;\sigma),\rho z_t + \sigma \varepsilon_{t+1};\sigma)} \\ c(k_t, z_t; \sigma) + k(k_t, z_t; \sigma) - e^{z_t} k_t^{\alpha} \end{array} \right] = \left[\begin{array}{c} 0 \\ 0 \end{array} \right]$$

- Note that:

$$F(k_t, z_t; \sigma) = \mathcal{H}(c_t, c_{t+1}, k_t, k_{t+1}, z_t; \sigma)$$
$$= \mathcal{H}(c(k_t, z_t; \sigma), c(k(k_t, z_t; \sigma), z_{t+1}; \sigma), k_t, k(k_t, z_t; \sigma), z_t; \sigma)$$

- I will use $\mathcal{H}_i$ to represent the partial derivative of $\mathcal{H}$ with respect to the i component and drop the evaluation at the steady state of the functions when we do not need it.

Zeroth-Order Approximation

- First, we evaluate $\sigma = 0$:

$$F\left(k_t, 0; 0\right) = 0$$

- Steady state:

$$\frac{1}{c} = \beta \frac{\alpha k^{\alpha-1}}{c}$$

or

$$1 = \alpha \beta k^{\alpha-1}$$

- Then:

$$c = c\left(k, 0; 0\right) = \left(\alpha\beta\right)^{\frac{\alpha}{1-\alpha}} - \left(\alpha\beta\right)^{\frac{1}{1-\alpha}}$$

$$k = k\left(k, 0; 0\right) = \left(\alpha\beta\right)^{\frac{1}{1-\alpha}}$$

First-Order Approximation

- We take derivatives of $F\left(k_t, z_t; \sigma\right)$ around $k, 0$, and 0.
- With respect to k_t:
$$F_k\left(k, 0; 0\right) = 0$$
- With respect to z_t:
$$F_z\left(k, 0; 0\right) = 0$$
- With respect to σ:
$$F_\sigma\left(k, 0; 0\right) = 0$$

Solving the System I

- Remember that:

$$F\left(k_t, z_t; \sigma\right)$$
$$= \mathcal{H}\left(c\left(k_t, z_t; \sigma\right), c\left(k\left(k_t, z_t; \sigma\right), z_{t+1}; \sigma\right), k_t, k\left(k_t, z_t; \sigma\right), z_t; \sigma\right) = 0$$

- Because $F\left(k_t, z_t; \sigma\right)$ must be equal to zero for any possible values of k_t, z_t, and σ, the derivatives of any order of F must also be zero.

- Then:

$$F_k\left(k, 0; 0\right) = \mathcal{H}_1 c_k + \mathcal{H}_2 c_k k_k + \mathcal{H}_3 + \mathcal{H}_4 k_k = 0$$
$$F_z\left(k, 0; 0\right) = \mathcal{H}_1 c_z + \mathcal{H}_2\left(c_k k_z + c_k \rho\right) + \mathcal{H}_4 k_z + \mathcal{H}_5 = 0$$
$$F_\sigma\left(k, 0; 0\right) = \mathcal{H}_1 c_\sigma + \mathcal{H}_2\left(c_k k_\sigma + c_\sigma\right) + \mathcal{H}_4 k_\sigma + \mathcal{H}_6 = 0$$

Solving the System II

- A quadratic system:

$$F_k(k,0;0) = \mathcal{H}_1 c_k + \mathcal{H}_2 c_k k_k + \mathcal{H}_3 + \mathcal{H}_4 k_k = 0$$
$$F_z(k,0;0) = \mathcal{H}_1 c_z + \mathcal{H}_2 (c_k k_z + c_k \rho) + \mathcal{H}_4 k_z + \mathcal{H}_5 = 0$$

of 4 equations on 4 unknowns: c_k, c_z, k_k, and k_z.

- Procedures to solve quadratic systems:
  1. Blanchard and Kahn (1980).
  2. Uhlig (1999).
  3. Sims (2000).
  4. Klein (2000).
- All of them equivalent.
- Why quadratic? Stable and unstable manifold.

Solving the System III

- Also, note that:

$$F_{\sigma}(k, 0; 0) = \mathcal{H}_1 c_{\sigma} + \mathcal{H}_2 (c_k k_{\sigma} + c_{\sigma}) + \mathcal{H}_4 k_{\sigma} + \mathcal{H}_6 = 0$$

  is a linear, and homogeneous system in c_{σ} and k_{σ}.

- Hence:

$$c_{\sigma} = k_{\sigma} = 0$$

- This means the system is certainty equivalent.
- Interpretation⇒no precautionary behavior.
- Difference between risk-aversion and precautionary behavior. Leland (1968), Kimball (1990).
- Risk-aversion depends on the second derivative (concave utility).
- Precautionary behavior depends on the third derivative (convex marginal utility).

Comparison with LQ and Linearization

- After Kydland and Prescott (1982) a popular method to solve economic models has been to find a LQ approximation of the objective function of the agents.

- Close relative: linearization of equilibrium conditions.

- When properly implemented linearization, LQ, and first-order perturbation are equivalent.

- Advantages of perturbation:

  1. Theorems.

  2. Higher-order terms.

Some Further Comments

- Note how we have used a version of the implicit-function theorem.
- Important tool in economics.
- Also, we are using the Taylor theorem to approximate the policy function.
- Alternatives?

Second-Order Approximation

- We take second-order derivatives of $F\left(k_t, z_t; \sigma\right)$ around $k, 0$, and 0:

$$
\begin{aligned}
F_{kk}(k,0;0) &= 0 \\
F_{kz}(k,0;0) &= 0 \\
F_{k\sigma}(k,0;0) &= 0 \\
F_{zz}(k,0;0) &= 0 \\
F_{z\sigma}(k,0;0) &= 0 \\
F_{\sigma\sigma}(k,0;0) &= 0
\end{aligned}
$$

- Remember Young's theorem!
- We substitute the coefficients that we already know.
- A linear system of 12 equations on 12 unknowns. Why linear?
- Cross-terms $k\sigma$ and $z\sigma$ are zero.
- Conjecture on all the terms with odd powers of σ.

Correction for Risk

- We have a term in σ^2.
- Captures precautionary behavior.
- We do not have certainty equivalence any more!
- Important advantage of second-order approximation.
- Changes ergodic distribution of states.

Higher-Order Terms

- We can continue the iteration for as long as we want.
- Great advantage of procedure: it is recursive!
- Often, a few iterations will be enough.
- The level of accuracy depends on the goal of the exercise:
  1. Welfare analysis: Kim and Kim (2001).
  2. Empirical strategies: Fernández-Villaverde, Rubio-Ramírez, and Santos (2006).

A Numerical Example

Parameter	β	α	ρ	σ
Value	0.99	0.33	0.95	0.01

- Steady State: $c = 0.388069 \quad k = 0.1883$
- First-order terms:

$$c_k(k,0;0) = 0.680101 \quad k_k(k,0;0) = 0.33$$
$$c_z(k,0;0) = 0.388069 \quad k_z(k,0;0) = 0.1883$$

- Second-order terms:

$$c_{kk}(k,0;0) = -2.41990 \quad k_{kk}(k,0;0) = -1.1742$$
$$c_{kz}(k,0;0) = 0.680099 \quad k_{kz}(k,0;0) = 0.33$$
$$c_{zz}(k,0;0) = 0.388064 \quad k_{zz}(k,0;0) = 0.1883$$
$$c_{\sigma^2}(k,0;0) \simeq 0 \quad k_{\sigma^2}(k,0;0) \simeq 0$$

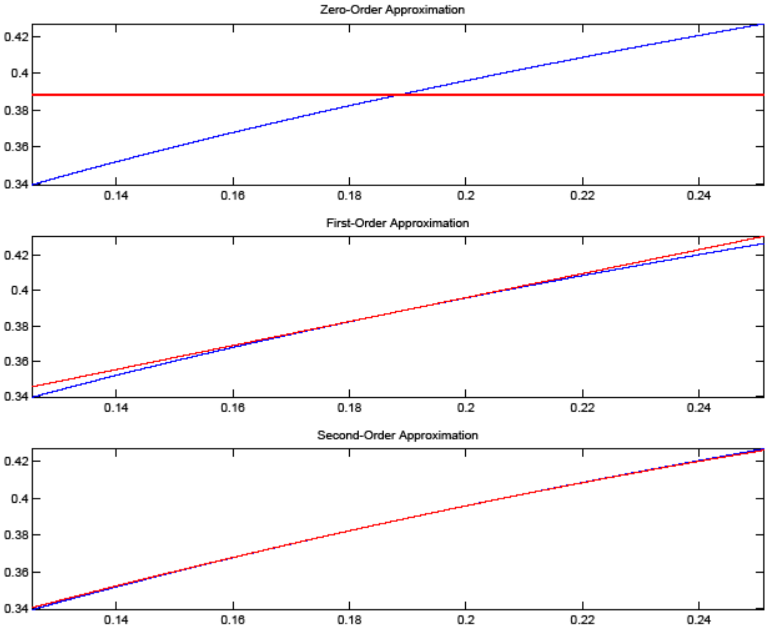
- $c_\sigma(k,0;0) = k_\sigma(k,0;0) = c_{k\sigma}(k,0;0) = k_{k\sigma}(k,0;0) = c_{z\sigma}(k,0;0) = k_{z\sigma}(k,0;0) = 0.$

Comparison

$$
\begin{aligned}
c_t &= 0.6733 e^{z_t} k_t^{0.33} \\
c_t &\simeq 0.388069 + 0.680101 \left(k_t - k\right) + 0.388069 z_t \\
&- \frac{2.41990}{2} \left(k_t - k\right)^2 + 0.680099 \left(k_t - k\right) z_t + \frac{0.388064}{2} z_t^2
\end{aligned}
$$

and:

$$
\begin{aligned}
k_{t+1} &= 0.3267 e^{z_t} k_t^{0.33} \\
k_{t+1} &\simeq 0.1883 + 0.33 \left(k_t - k\right) + 0.1883 z_t \\
&- \frac{1.1742}{2} \left(k_t - k\right)^2 + 0.33 \left(k_t - k\right) z_t + \frac{0.1883}{2} z_t^2
\end{aligned}
$$


Zero-Order Approximation
First-Order Approximation
Second-Order Approximation
0.42
0.4
0.38
0.36
0.34
0.14
0.16
0.18
0.2
0.22
0.24

A Computer

- In practice you do all this approximations with a computer:

  1. First-, second-, and third-order: `Matlab` and `Dynare`.
  2. Higher-order: `Mathematica`, `Dynare++`, `Fortran` code by Jinn and Judd.

- Burden: analytical derivatives.

- Why are numerical derivatives a bad idea?

- Alternatives: automatic differentiation?

Local Properties of the Solution

- Perturbation is a local method.
- It approximates the solution around the deterministic steady state of the problem.
- It is valid within a radius of convergence.
- What is the radius of convergence of a power series around x? An $r \in \mathbb{R}_{+}^{\infty}$ such that $\forall x'$, $|x' - z| < r$, the power series of x' will converge.

A Remarkable Result from Complex Analysis

The radius of convergence is always equal to the distance from the center to the nearest point where the policy function has a (non-removable) singularity. If no such point exists then the radius of convergence is infinite.

- Singularity here refers to poles, fractional powers, and other branch powers or discontinuities of the functional or its derivatives.

Remarks

- Intuition of the theorem: holomorphic functions are analytic.
- Distance is in the complex plane.
- Often, we can check numerically that perturbations have good non local behavior.
- However: problem with boundaries.

Non Local Accuracy Test

- Proposed by Judd (1992) and Judd and Guu (1997).

- Given the Euler equation:

$$\frac{1}{c^i\left(k_t, z_t\right)} = \mathbb{E}_t\left(\frac{\alpha e^{z_{t+1}} k^i\left(k_t, z_t\right)^{\alpha-1}}{c^i\left(k^i\left(k_t, z_t\right), z_{t+1}\right)}\right)$$

we can define:

$$EE^i\left(k_t, z_t\right) \equiv 1 - c^i\left(k_t, z_t\right)\mathbb{E}_t\left(\frac{\alpha e^{z_{t+1}} k^i\left(k_t, z_t\right)^{\alpha-1}}{c^i\left(k^i\left(k_t, z_t\right), z_{t+1}\right)}\right)$$

- Units of reporting.

- Interpretation.

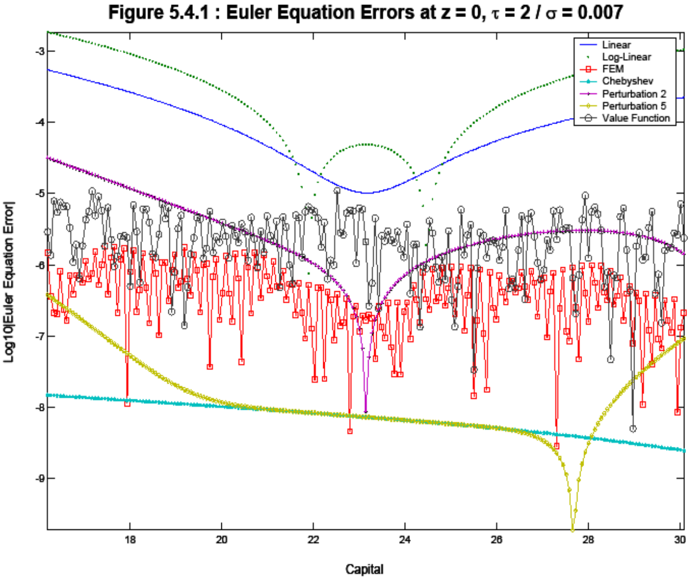


Figure 5.4.1 : Euler Equation Errors at z = 0, τ = 2 / σ = 0.007
Linear
Log-Linear
FEM
Chebyshev
Perturbation 2
Perturbation 5
Value Function
Log10|Euler Equation Error|
-3
-4
-5
-6
-7
-8
-9
18
20
22
24
26
28
30
Capital

The General Case

- Most of previous argument can be easily generalized.

- The set of equilibrium conditions of many DSGE models can be written as (note recursive notation)

$$\mathbb{E}_t \mathcal{H}(y, y', x, x') = 0,$$

where y_t is a $n_y \times 1$ vector of controls and x_t is a $n_x \times 1$ vector of states.

- Define $n = n_x + n_y$.

- Then $\mathcal{H}$ maps $R^{n_y} \times R^{n_y} \times R^{n_x} \times R^{n_x}$ into R^n.

Partitioning the State Vector

- The state vector x_t can be partitioned as $x = [x_1; x_2]^t$.
- x_1 is a $(n_x - n_\epsilon) \times 1$ vector of endogenous state variables.
- x_2 is a $n_\epsilon \times 1$ vector of exogenous state variables.
- Why do we want to partition the state vector?

Exogenous Stochastic Process

$$x_2' = \Lambda x_2 + \sigma \eta_\epsilon \epsilon'$$

- Process with 3 parts:
  1. The deterministic component Λx_2:
     1. Λ is a $n_\epsilon \times n_\epsilon$ matrix, with all eigenvalues with modulus less than one.
     2. More general: $x_2' = \Gamma(x_2) + \sigma \eta_\epsilon \epsilon'$, where Γ is a non-linear function satisfying that all eigenvalues of its first derivative evaluated at the non-stochastic steady state lie within the unit circle.
  2. The scaled innovation $\eta_\epsilon \epsilon'$ where:
     1. η_ϵ is a known $n_\epsilon \times n_\epsilon$ matrix.
     2. ϵ is a $n_\epsilon \times 1$ i.i.d innovation with bounded support, zero mean, and variance/covariance matrix I.
  3. The perturbation parameter σ.
- We can accommodate very general structures of x_2 through changes in the definition of the state space: i.e. stochastic volatility.
- Note we do not impose Gaussianity.

The Perturbation Parameter

- The scalar $\sigma \geq 0$ is the perturbation parameter.

- If we set $\sigma = 0$ we have a deterministic model.

- Important: there is only ONE perturbation parameter. The matrix η_ϵ takes account of relative sizes of different shocks.

- Why bounded support? Samuelson (1970) and Jin and Judd (2002).

Solution of the Model

- The solution to the model is of the form:

$$y = g(x;\sigma)$$
$$x' = h(x;\sigma) + \sigma\eta\epsilon'$$

where g maps $R^{n_x} \times R^{+}$ into R^{n_y} and h maps $R^{n_x} \times R^{+}$ into R^{n_x}.

- The matrix η is of order $n_x \times n_\epsilon$ and is given by:

$$\eta = \begin{bmatrix} \emptyset \\ \eta_\epsilon \end{bmatrix}$$

Perturbation

- We wish to find a perturbation approximation of the functions g and h around the non-stochastic steady state, $x_t = \bar{x}$ and $\sigma = 0$.

- We define the non-stochastic steady state as vectors $(\bar{x}, \bar{y})$ such that:

$$\mathcal{H}(\bar{y}, \bar{y}, \bar{x}, \bar{x}) = 0.$$

- Note that $\bar{y} = g(\bar{x}; 0)$ and $\bar{x} = h(\bar{x}; 0)$. This is because, if $\sigma = 0$, then $\mathbb{E}_t \mathcal{H} = \mathcal{H}$.

Plugging-in the Proposed Solution

- Substituting the proposed solution, we define:

$$F(x;\sigma) \equiv \mathbb{E}_t \mathcal{H}(g(x;\sigma), g(h(x;\sigma) + \eta\sigma\epsilon', \sigma), x, h(x;\sigma) + \eta\sigma\epsilon') = 0$$

- Since $F(x;\sigma) = 0$ for any values of x and σ, the derivatives of any order of F must also be equal to zero.

- Formally:

$$F_{x^k\sigma^j}(x;\sigma) = 0 \quad \forall x, \sigma, j, k,$$

where $F_{x^k\sigma^j}(x, \sigma)$ denotes the derivative of F with respect to x taken k times and with respect to σ taken j times.

First-Order Approximation

- We look for approximations to g and h around $(x, \sigma) = (\bar{x}, 0)$:

$$\begin{aligned} g(x;\sigma) &= g(\bar{x};0) + g_x(\bar{x};0)(x-\bar{x}) + g_\sigma(\bar{x};0)\sigma \\ h(x;\sigma) &= h(\bar{x};0) + h_x(\bar{x};0)(x-\bar{x}) + h_\sigma(\bar{x};0)\sigma \end{aligned}$$

- As explained earlier,

$$g(\bar{x};0) = \bar{y}$$

  and

$$h(\bar{x};0) = \bar{x}.$$

- The four unknown coefficients of the first-order approximation to g and h are found by using:

$$F_x(\bar{x};0) = 0$$

  and

$$F_\sigma(\bar{x};0) = 0$$

- Before doing so, I need to introduce the tensor notation.

Tensors

- General trick from physics.
- An n^{th}-rank tensor in a m-dimensional space is an operator that has n indices and m^n components and obeys certain transformation rules.
- $[\mathcal{H}_y]^i_\alpha$ is the (i, α) element of the derivative of $\mathcal{H}$ with respect to y:
  1. The derivative of $\mathcal{H}$ with respect to y is an $n \times n_y$ matrix.
  2. Thus, $[\mathcal{H}_y]^i_\alpha$ is the element of this matrix located at the intersection of the i-th row and α-th column.
  3. Thus, $[\mathcal{H}_y]^i_\alpha [g_x]^\alpha_\beta [h_x]^\beta_j = \sum_{\alpha=1}^{n_y} \sum_{\beta=1}^{n_x} \frac{\partial \mathcal{H}^i}{\partial y^\alpha} \frac{\partial g^\alpha}{\partial x^\beta} \frac{\partial h^\beta}{\partial x^j}$.
- $[\mathcal{H}_{y'y'}]^i_{\alpha\gamma}$:
  1. $\mathcal{H}_{y'y'}$ is a three dimensional array with n rows, n_y columns, and n_y pages.
  2. Then $[\mathcal{H}_{y'y'}]^i_{\alpha\gamma}$ denotes the element of $\mathcal{H}_{y'y'}$ located at the intersection of row i, column α and page γ.

Solving the System I

- g_x and h_x can be found as the solution to the system:

$$\begin{aligned} [F_x(\bar{x};0)]^i_j &= [\mathcal{H}_{y'}]^i_\alpha [g_x]^\alpha_\beta [h_x]^\beta_j + [\mathcal{H}_y]^i_\alpha [g_x]^\alpha_j + [\mathcal{H}_{x'}]^i_\beta [h_x]^\beta_j + [\mathcal{H}_x]^i_j = \\ i &= 1,\ldots,n; \quad j,\beta = 1,\ldots,n_x; \quad \alpha = 1,\ldots,n_y \end{aligned}$$

- Note that the derivatives of $\mathcal{H}$ evaluated at $(y, y', x, x') = (\bar{y}, \bar{y}, \bar{x}, \bar{x})$ are known.

- Then, we have a system of $n \times n_x$ quadratic equations in the $n \times n_x$ unknowns given by the elements of g_x and h_x.

- We can solve with a standard quadratic matrix equation solver.

Solving the System II

- g_σ and h_σ are identified as the solution to the following n equations:

$$[F_\sigma(\bar{x};0)]^i =$$

$$\mathbb{E}_t\{[\mathcal{H}_{y'}]^i_\alpha [g_x]^\alpha_\beta [h_\sigma]^\beta + [\mathcal{H}_{y'}]^i_\alpha [g_x]^\alpha_\beta [\eta]^\beta_\phi [\epsilon']^\phi + [\mathcal{H}_{y'}]^i_\alpha [g_\sigma]^\alpha$$

$$+[\mathcal{H}_y]^i_\alpha [g_\sigma]^\alpha + [\mathcal{H}_{x'}]^i_\beta [h_\sigma]^\beta + [\mathcal{H}_{x'}]^i_\beta [\eta]^\beta_\phi [\epsilon']^\phi\}$$

$$i = 1, \ldots, n; \quad \alpha = 1, \ldots, n_y; \quad \beta = 1, \ldots, n_x; \quad \phi = 1, \ldots, n_\epsilon.$$

- Then:

$$[F_\sigma(\bar{x};0)]^i = [\mathcal{H}_{y'}]^i_\alpha [g_x]^\alpha_\beta [h_\sigma]^\beta + [\mathcal{H}_{y'}]^i_\alpha [g_\sigma]^\alpha + [\mathcal{H}_y]^i_\alpha [g_\sigma]^\alpha + [f_{x'}]^i_\beta [h_\sigma]^\beta = 0;$$

$$i = 1, \ldots, n; \quad \alpha = 1, \ldots, n_y; \quad \beta = 1, \ldots, n_x; \quad \phi = 1, \ldots, n_\epsilon.$$

- Certainty equivalence: this equation is linear and homogeneous in g_σ and h_σ. Thus, if a unique solution exists, it must satisfy:

$$\begin{aligned} h_\sigma &\neq 0 \\ g_\sigma &= 0 \end{aligned}$$

Second-Order Approximation I

The second-order approximations to g around $(x;\sigma) = (\bar{x};0)$ is

$$
\begin{aligned}
[g(x;\sigma)]^i \quad = \quad & [g(\bar{x};0)]^i + [g_x(\bar{x};0)]^i_a[(x-\bar{x})]_a + [g_\sigma(\bar{x};0)]^i[\sigma] \\
& + \frac{1}{2}[g_{xx}(\bar{x};0)]^i_{ab}[(x-\bar{x})]_a[(x-\bar{x})]_b \\
& + \frac{1}{2}[g_{x\sigma}(\bar{x};0)]^i_a[(x-\bar{x})]_a[\sigma] \\
& + \frac{1}{2}[g_{\sigma x}(\bar{x};0)]^i_a[(x-\bar{x})]_a[\sigma] \\
& + \frac{1}{2}[g_{\sigma\sigma}(\bar{x};0)]^i[\sigma][\sigma]
\end{aligned}
$$

where $i = 1, \ldots, n_y$, $a, b = 1, \ldots, n_x$, and $j = 1, \ldots, n_x$.

Second-Order Approximation II

The second-order approximations to h around $(x;\sigma) = (\bar{x};0)$ is

$$\begin{aligned}
[h(x;\sigma)]^j &= [h(\bar{x};0)]^j + [h_x(\bar{x};0)]^j_a[(x-\bar{x})]_a + [h_\sigma(\bar{x};0)]^j[\sigma] \\
&+\frac{1}{2}[h_{xx}(\bar{x};0)]^j_{ab}[(x-\bar{x})]_a[(x-\bar{x})]_b \\
&+\frac{1}{2}[h_{x\sigma}(\bar{x};0)]^j_a[(x-\bar{x})]_a[\sigma] \\
&+\frac{1}{2}[h_{\sigma x}(\bar{x};0)]^j_a[(x-\bar{x})]_a[\sigma] \\
&+\frac{1}{2}[h_{\sigma\sigma}(\bar{x};0)]^j[\sigma][\sigma],
\end{aligned}$$

where $i = 1, \ldots, n_y$, $a, b = 1, \ldots, n_x$, and $j = 1, \ldots, n_x$.

Second-Order Approximation III

- The unknowns of these expansions are $[g_{xx}]^i_{ab}$, $[g_{x\sigma}]^i_a$, $[g_{\sigma x}]^i_a$, $[g_{\sigma\sigma}]^i$, $[h_{xx}]^j_{ab}$, $[h_{x\sigma}]^j_a$, $[h_{\sigma x}]^j_a$, $[h_{\sigma\sigma}]^j$.

- These coefficients can be identified by taking the derivative of $F(x;\sigma)$ with respect to x and σ twice and evaluating them at $(x;\sigma) = (\bar{x};0)$.

- By the arguments provided earlier, these derivatives must be zero.

Solving the System I

We use $F_{xx}(\bar{x};0)$ to identify $g_{xx}(\bar{x};0)$ and $h_{xx}(\bar{x};0)$:

$$\begin{gathered}
[F_{xx}(\bar{x};0)]^i_{jk} = \\
\left([\mathcal{H}_{y'y'}]^i_{\alpha\gamma}[g_x]^\gamma_\delta[h_x]^\delta_k + [\mathcal{H}_{y'y}]^i_{\alpha\gamma}[g_x]^\gamma_k + [\mathcal{H}_{y'x'}]^i_{\alpha\delta}[h_x]^\delta_k + [\mathcal{H}_{y'x}]^i_{\alpha k}\right)[g_x]^\alpha_\beta[h_x]^\beta_j \\
+[\mathcal{H}_{y'}]^i_\alpha[g_{xx}]^\alpha_{\beta\delta}[h_x]^\delta_k[h_x]^\beta_j + [\mathcal{H}_{y'}]^i_\alpha[g_x]^\alpha_\beta[h_{xx}]^\beta_{jk} \\
+\left([\mathcal{H}_{yy'}]^i_{\alpha\gamma}[g_x]^\gamma_\delta[h_x]^\delta_k + [\mathcal{H}_{yy}]^i_{\alpha\gamma}[g_x]^\gamma_k + [\mathcal{H}_{yx'}]^i_{\alpha\delta}[h_x]^\delta_k + [\mathcal{H}_{yx}]^i_{\alpha k}\right)[g_x]^\alpha_j \\
+[\mathcal{H}_y]^i_\alpha[g_{xx}]^\alpha_{jk} \\
+\left([\mathcal{H}_{x'y'}]^i_{\beta\gamma}[g_x]^\gamma_\delta[h_x]^\delta_k + [\mathcal{H}_{x'y}]^i_{\beta\gamma}[g_x]^\gamma_k + [\mathcal{H}_{x'x'}]^i_{\beta\delta}[h_x]^\delta_k + [\mathcal{H}_{x'x}]^i_{\beta k}\right)[h_x]^\beta_j \\
+[\mathcal{H}_{x'}]^i_\beta[h_{xx}]^\beta_{jk} \\
+[\mathcal{H}_{xy'}]^i_{j\gamma}[g_x]^\gamma_\delta[h_x]^\delta_k + [\mathcal{H}_{xy}]^i_{j\gamma}[g_x]^\gamma_k + [\mathcal{H}_{xx'}]^i_{j\delta}[h_x]^\delta_k + [\mathcal{H}_{xx}]^i_{jk} = 0; \\
i = 1,\dots n, \quad j,k,\beta,\delta = 1,\dots n_x; \quad \alpha,\gamma = 1,\dots n_y.
\end{gathered}$$

Solving the System II

- We know the derivatives of $\mathcal{H}$.
- We also know the first derivatives of g and h evaluated at $(y, y', x, x') = (\bar{y}, \bar{y}, \bar{x}, \bar{x})$.
- Hence, the above expression represents a system of $n \times n_x \times n_x$ linear equations in then $n \times n_x \times n_x$ unknowns elements of g_{xx} and h_{xx}.

Solving the System III

Similarly, $g_{\sigma\sigma}$ and $h_{\sigma\sigma}$ can be obtained by solving:

$$
\begin{aligned}
[F_{\sigma\sigma}(\bar{x};0)]^i &= [\mathcal{H}_{y'}]^i_\alpha [g_x]^\alpha_\beta [h_{\sigma\sigma}]^\beta \\
&+[\mathcal{H}_{y'y'}]^i_{\alpha\gamma}[g_x]^\gamma_\delta[\eta]^\delta_\xi[g_x]^\alpha_\beta[\eta]^\beta_\phi[I]^\phi_\xi \\
&+[\mathcal{H}_{y'x'}]^i_{\alpha\delta}[\eta]^\delta_\xi[g_x]^\alpha_\beta[\eta]^\beta_\phi[I]^\phi_\xi \\
&+[\mathcal{H}_{y'}]^i_\alpha[g_{xx}]^\alpha_{\beta\delta}[\eta]^\delta_\xi[\eta]^\beta_\phi[I]^\phi_\xi + [\mathcal{H}_{y'}]^i_\alpha[g_{\sigma\sigma}]^\alpha \\
&+[\mathcal{H}_y]^i_\alpha[g_{\sigma\sigma}]^\alpha + [\mathcal{H}_{x'}]^i_\beta[h_{\sigma\sigma}]^\beta \\
&+[\mathcal{H}_{x'y'}]^i_{\beta\gamma}[g_x]^\gamma_\delta[\eta]^\delta_\xi[\eta]^\beta_\phi[I]^\phi_\xi \\
&+[\mathcal{H}_{x'x'}]^i_{\beta\delta}[\eta]^\delta_\xi[\eta]^\beta_\phi[I]^\phi_\xi = 0; \\
i &= 1,\ldots,n;\alpha,\gamma = 1,\ldots,n_y;\beta,\delta = 1,\ldots,n_x;\phi,\xi = 1,\ldots,n_\epsilon
\end{aligned}
$$

a system of n linear equations in the n unknowns given by the elements of $g_{\sigma\sigma}$ and $h_{\sigma\sigma}$.

Cross Derivatives

- The cross derivatives $g_{x\sigma}$ and $h_{x\sigma}$ are zero when evaluated at $(\bar{x}, 0)$.
- Why? Write the system $F_{\sigma x}(\bar{x}; 0) = 0$ taking into account that all terms containing either g_{σ} or h_{σ} are zero at $(\bar{x}, 0)$.
- Then:

$$[F_{\sigma x}(\bar{x};0)]^i_j = [\mathcal{H}_{y'}]^i_\alpha [g_x]^\alpha_\beta [h_{\sigma x}]^\beta_j + [\mathcal{H}_{y'}]^i_\alpha [g_{\sigma x}]^\alpha_\gamma [h_x]^\gamma_j +$$
$$[\mathcal{H}_y]^i_\alpha [g_{\sigma x}]^\alpha_j + [\mathcal{H}_{x'}]^i_\beta [h_{\sigma x}]^\beta_j = 0;$$
$$i = 1, \ldots n; \quad \alpha = 1, \ldots, n_y; \quad \beta, \gamma, j = 1, \ldots, n_x.$$

  a system of $n \times n_x$ equations in the $n \times n_x$ unknowns given by the elements of $g_{\sigma x}$ and $h_{\sigma x}$.
- The system is homogeneous in the unknowns.
- Thus, if a unique solution exists, it is given by:

$$g_{\sigma x} = 0$$
$$h_{\sigma x} = 0$$

Structure of the Solution

- The perturbation solution of the model satisfies:

$$
\begin{aligned}
g_{\sigma}(\bar{x};0) &= 0 \\
h_{\sigma}(\bar{x};0) &= 0 \\
g_{x\sigma}(\bar{x};0) &= 0 \\
h_{x\sigma}(\bar{x};0) &= 0
\end{aligned}
$$

- Standard deviation only appears in:
  1. A constant term given by $\frac{1}{2}g_{\sigma\sigma}\sigma^2$ for the control vector y_t.
  2. The first $n_x - n_\epsilon$ elements of $\frac{1}{2}h_{\sigma\sigma}\sigma^2$.
- Correction for risk.
- Quadratic terms in endogenous state vector x_1.
- Those terms capture non-linear behavior.

Higher-Order Approximations

- We can iterate this procedure as many times as we want.
- We can obtain n-th order approximations.
- Problems:
  1. Existence of higher order derivatives (Santos, 1992).
  2. Numerical instabilities.
  3. Computational costs.

Erik Eady

It is not the process of linearization that limits insight.
It is the nature of the state that we choose to linearize about.

Change of Variables

- We approximated our solution in levels.
- We could have done it in logs.
- Why stop there? Why not in powers of the state variables?
- Judd (2002) has provided methods for changes of variables.
- We apply and extend ideas to the stochastic neoclassical growth model.

A General Transformation

- We look at solutions of the form:

$$
\begin{aligned}
c^{\mu} - c_0^{\mu} &= a\left(k^{\zeta} - k_0^{\zeta}\right) + bz \\
k'^{\gamma} - k_0^{\gamma} &= c\left(k^{\zeta} - k_0^{\zeta}\right) + dz
\end{aligned}
$$

- Note that:

1. If γ, ζ, and μ are 1, we get the linear representation.
2. As γ, ζ and μ tend to zero, we get the loglinear approximation.

Theory

- The first-order solution can be written as

$$f(x) \simeq f(a) + (x-a) f'(a)$$

- Expand $g(y) = h(f(X(y)))$ around $b = Y(a)$, where $X(y)$ is the inverse of $Y(x)$.

- Then:

$$g(y) = h(f(X(y))) = g(b) + g_\alpha(b) \left(Y^\alpha(x) - b^\alpha\right)$$

where $g_\alpha = h_A f_i^A X_\alpha^i$ comes from the application of the chain rule.

- From this expression it is easy to see that if we have computed the values of f_i^A, then it is straightforward to find g_α.

Coefficients Relation

- Remember that the linear solution is:

$$
\begin{aligned}
\left(k' - k_0\right) &= a_1 \left(k - k_0\right) + b_1 z \\
\left(l - l_0\right) &= c_1 \left(k - k_0\right) + d_1 z
\end{aligned}
$$

- Then we show that:

$a_3 = \frac{\gamma}{\zeta} k_0^{\gamma-\zeta} a_1$	$b_3 = \gamma k_0^{\gamma-1} b_1$
$c_3 = \frac{\mu}{\zeta} l_0^{\mu-1} k_0^{1-\zeta} c_1$	$d_3 = \mu l_0^{\mu-1} d_1$

Finding the Parameters

- Minimize over a grid the Euler Error.
- Some optimal results

Euler Equation Errors

γ	ζ	μ	*SEE*
1	1	1	0.0856279
0.986534	0.991673	2.47856	0.0279944

Sensitivity Analysis

- Different parameter values.

- Most interesting finding is when we change σ:

Optimal Parameters for different σ's

σ	γ	ζ	μ
0.014	0.98140	0.98766	2.47753
0.028	1.04804	1.05265	1.73209
0.056	1.23753	1.22394	0.77869

- A first-order approximation corrects for changes in variance!

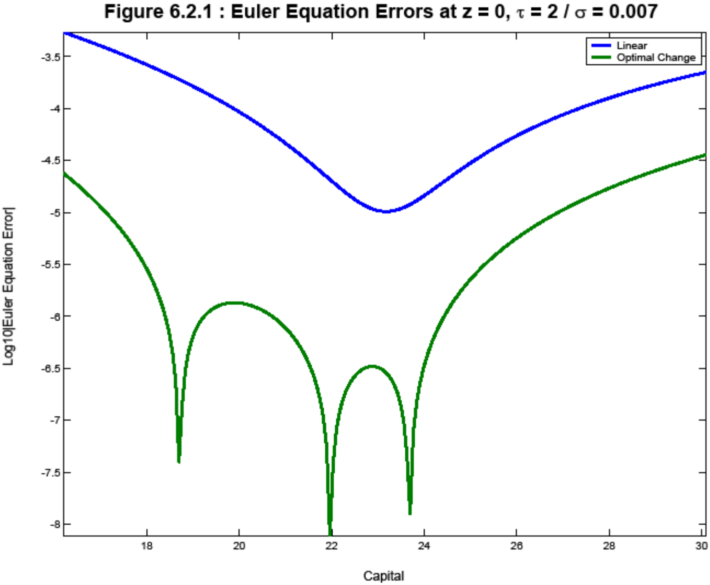


Figure 6.2.1 : Euler Equation Errors at z = 0, τ = 2 / σ = 0.007
Linear
Optimal Change
Log10|Euler Equation Error|
-3.5
-4
-4.5
-5
-5.5
-6
-6.5
-7
-7.5
-8
18
20
22
24
26
28
30
Capital

A Quasi-Optimal Approximation

- Sensitivity analysis reveals that for different parametrizations

$$\gamma \simeq \zeta$$

- This suggests the quasi-optimal approximation:

$$\begin{aligned} k'^{\gamma} - k_0^{\gamma} &= a_3 \left(k^{\gamma} - k_0^{\gamma}\right) + b_3 z \\ l^{\mu} - l_0^{\mu} &= c_3 \left(k^{\gamma} - k_0^{\gamma}\right) + d_3 z \end{aligned}$$

- If we define $\widehat{k} = k^{\gamma} - k_0^{\gamma}$ and $\widehat{l} = l^{\mu} - l_0^{\mu}$ we get:

$$\begin{aligned} \widehat{k}' &= a_3 \widehat{k} + b_3 z \\ \widehat{l} &= c_3 \widehat{k} + d_3 z \end{aligned}$$

- Linear system:

  1. Use for analytical study.
  2. Use for estimation with a Kalman Filter.

Perturbing the Value Function

- We worked with the equilibrium conditions of the model.
- Sometimes we may want to perform a perturbation on the value function formulation of the problem.
- Possible reasons:
  1. Gain insight.
  2. Difficulty in using equilibrium conditions.
  3. Evaluate welfare.
  4. Initial guess for VFI.

Basic Problem

- Imagine that we have:

$$V(k_t, z_t) = \max_{c_t} \left[(1-\beta) \frac{c_t^{1-\gamma}}{1-\gamma} + \beta \mathbb{E}_t V(k_{t+1}, z_{t+1}) \right]$$

$$\text{s.t. } c_t + k_{t+1} = e^{z_t} k_t^{\theta} + (1-\delta) k_t$$

$$z_t = \lambda z_{t-1} + \sigma \varepsilon_t,\ \varepsilon_t \sim \mathcal{N}(0,1)$$

- Write it as:

$$V(k_t, z_t; \chi) = \max_{c_t} \left[(1-\beta) \frac{c_t^{1-\gamma}}{1-\gamma} + \beta \mathbb{E}_t V(k_{t+1}, z_{t+1}; \chi) \right]$$

$$\text{s.t. } c_t + k_{t+1} = e^{z_t} k_t^{\theta} + (1-\delta) k_t$$

$$z_t = \lambda z_{t-1} + \chi \sigma \varepsilon_t,\ \varepsilon_t \sim \mathcal{N}(0,1)$$

Alternative

- Another way to write the value function is:

$$V(k_t, z_t; \chi) = \max_{c_t} \left[\begin{array}{c} (1-\beta)\frac{c_t^{1-\gamma}}{1-\gamma} + \\ \beta \mathbb{E}_t V\left(e^{z_t} k_t^{\theta} + (1-\delta) k_t - c_t, \lambda z_t + \chi \sigma \varepsilon_{t+1}; \chi\right) \end{array} \right]$$

- This form makes the dependences in the next period states explicit.

- The solution of this problem is value function $V(k_t, z_t; \chi)$ and a policy function for consumption $c(k_t, z_t; \chi)$.

Expanding the Value Function

The second-order Taylor approximation of the value function around the deterministic steady state $(k_{ss}, 0; 0)$ is:

$$V(k_t, z_t; \chi) \simeq$$

$$V_{ss} + V_{1,ss}(k_t - k_{ss}) + V_{2,ss} z_t + V_{3,ss}\chi$$

$$+\frac{1}{2}V_{11,ss}(k_t - k_{ss})^2 + \frac{1}{2}V_{12,ss}(k_t - k_{ss}) z_t + \frac{1}{2}V_{13,ss}(k_t - k_{ss})\chi$$

$$+\frac{1}{2}V_{21,ss} z_t (k_t - k_{ss}) + \frac{1}{2}V_{22,ss} z_t^2 + \frac{1}{2}V_{23,ss} z_t \chi$$

$$+\frac{1}{2}V_{31,ss}\chi(k_t - k_{ss}) + \frac{1}{2}V_{32,ss}\chi z_t + \frac{1}{2}V_{33,ss}\chi^2$$

where

$$\begin{aligned} V_{ss} &= V(k_{ss}, 0; 0) \\ V_{i,ss} &= V_i(k_{ss}, 0; 0) \text{ for } i = \{1, 2, 3\} \\ V_{ij,ss} &= V_{ij}(k_{ss}, 0; 0) \text{ for } i, j = \{1, 2, 3\} \end{aligned}$$

Expanding the Value Function

- By certainty equivalence, we will show below that:

$$V_{3,ss} = V_{13,ss} = V_{23,ss} = 0$$

- Taking advantage of the equality of cross-derivatives, and setting $\chi = 1$, which is just a normalization:

$$\begin{aligned} V\left(k_t, z_t; 1\right) \simeq \; & V_{ss} + V_{1,ss}\left(k_t - k_{ss}\right) + V_{2,ss} z_t \\ & + \frac{1}{2} V_{11,ss}\left(k_t - k_{ss}\right)^2 + \frac{1}{2} V_{22,ss} z_{tt}^2 \\ & + V_{12,ss}\left(k_t - k_{ss}\right) z + \frac{1}{2} V_{33,ss} \end{aligned}$$

- Note that $V_{33,ss} \neq 0$, a difference from the standard linear-quadratic approximation to the utility functions.

Expanding the Consumption Function

- The policy function for consumption can be expanded as:

$$c_t = c\left(k_t, z_t; \chi\right) \simeq c_{ss} + c_{1,ss}\left(k_t - k_{ss}\right) + c_{2,ss} z_t + c_{3,ss}\chi$$

where:

$$\begin{aligned} c_{1,ss} &= c_1\left(k_{ss}, 0; 0\right) \\ c_{2,ss} &= c_2\left(k_{ss}, 0; 0\right) \\ c_{3,ss} &= c_3\left(k_{ss}, 0; 0\right) \end{aligned}$$

- Since the first derivatives of the consumption function only depend on the first and second derivatives of the value function, we must have $c_{3,ss} = 0$ (precautionary consumption depends on the third derivative of the value function, Kimball, 1990).

Linear Components of the Value Function

- To find the linear approximation to the value function, we take derivatives of the value function with respect to controls (c_t), states (k_t, z_t), and the perturbation parameter χ.

- Notation:

1. $V_{i,t}$: derivative of the value function with respect to its i-th argument, evaluated in $(k_t, z_t; \chi)$.
2. $V_{i,ss}$: derivative evaluated in the steady state, $(k_{ss}, 0; 0)$.
3. We follow the same notation for higher-order (cross-) derivatives.

Derivatives

- Derivative with respect to c_t:

$$(1-\beta)\, c_t^{-\gamma} - \beta \mathbb{E}_t V_{1,t+1} = 0$$

- Derivative with respect to k_t:

$$V_{1,t} = \beta \mathbb{E}_t V_{1,t+1} \left(\theta e^{z_t} k_t^{\theta-1} + 1 - \delta \right)$$

- Derivative with respect to z_t:

$$V_{2,t} = \beta \mathbb{E}_t \left[V_{1,t+1} e^{z_t} k_t^{\theta} + V_{2,t+1} \lambda \right]$$

- Derivative with respect to χ:

$$V_{3,t} = \beta \mathbb{E}_t \left[V_{2,t+1} \sigma \varepsilon_{t+1} + V_{3,t+1} \right]$$

- In the last three derivatives, we apply the envelope theorem to eliminate the derivatives of consumption with respect to k_t, z_t, and χ.

System of Equations I

Now, we have the system:

$$c_t + k_{t+1} = e^{z_t} k_t^{\theta} + (1-\delta)\, k_t$$

$$V(k_t, z_t; \chi) = (1-\beta)\frac{c_t^{1-\gamma}}{1-\gamma} + \beta \mathbb{E}_t V(k_{t+1}, z_{t+1}; \chi)$$

$$(1-\beta)\, c_t^{-\gamma} - \beta \mathbb{E}_t V_{1,t+1} = 0$$

$$V_{1,t} = \beta \mathbb{E}_t V_{1,t+1}\left(\theta e^{z_t} k_t^{\theta-1} + 1 - \delta\right)$$

$$V_{2,t} = \beta \mathbb{E}_t \left[V_{1,t+1} e^{z_t} k_t^{\theta} + V_{2,t+1}\lambda\right]$$

$$V_{3,t} = \beta \mathbb{E}_t \left[V_{2,t+1}\sigma\varepsilon_{t+1} + V_{3,t+1}\right]$$

$$z_t = \lambda z_{t-1} + \chi\sigma\varepsilon_t$$

System of Equations II

If we set $\chi = 0$ and compute the steady state, we get a system of six equations on six unknowns, c_{ss}, k_{ss}, V_{ss}, $V_{1,ss}$, $V_{2,ss}$, and $V_{3,ss}$:

$$c_{ss} + \delta k_{ss} = k_{ss}^{\theta}$$

$$V_{ss} = (1-\beta)\frac{c_{ss}^{1-\gamma}}{1-\gamma} + \beta V_{ss}$$

$$(1-\beta)\, c_{ss}^{-\gamma} - \beta V_{1,ss} = 0$$

$$V_{1,ss} = \beta V_{1,ss}\left(\theta k_{ss}^{\theta-1} + 1 - \delta\right)$$

$$V_{2,ss} = \beta\left[V_{1,ss} k_{ss}^{\theta} + V_{2,ss}\lambda\right]$$

$$V_{3,ss} = \beta V_{3,ss}$$

- From the last equation: $V_{3,ss} = 0$.
- From the second equation: $V_{ss} = \frac{c_{ss}^{1-\gamma}}{1-\gamma}$.
- From the third equation: $V_{1,ss} = \frac{1-\beta}{\beta} c_{ss}^{-\gamma}$.

System of Equations III

- After cancelling redundant terms:

$$c_{ss} + \delta k_{ss} = k_{ss}^{\theta}$$
$$1 = \beta \left(\theta k_{ss}^{\theta-1} + 1 - \delta \right)$$
$$V_{2,ss} = \beta \left[V_{1,ss} k_{ss}^{\theta} + V_{2,ss} \lambda \right]$$

- Then:

$$k_{ss} = \left[\frac{1}{\theta} \left(\frac{1}{\beta} - 1 + \delta \right) \right]^{\frac{1}{\theta-1}}$$
$$c_{ss} = k_{ss}^{\theta} - \delta k_{ss}$$
$$V_{2,ss} = \frac{1-\beta}{1-\beta\lambda} k_{ss}^{\theta} c_{ss}^{-\gamma}$$

- $V_{1,ss} > 0$ and $V_{2,ss} > 0$, as predicted by theory.

Quadratic Components of the Value Function

From the previous derivations, we have:

$$(1-\beta)\, c\,(k_t, z_t; \chi)^{-\gamma} - \beta \mathbb{E}_t V_{1,t+1} = 0$$

$$V_{1,t} = \beta \mathbb{E}_t V_{1,t+1} \left(\theta e^{z_t} k_t^{\theta-1} + 1 - \delta\right)$$

$$V_{2,t} = \beta \mathbb{E}_t \left[V_{1,t+1} e^{z_t} k_t^{\theta} + V_{2,t+1} \lambda\right]$$

$$V_{3,t} = \beta \mathbb{E}_t \left[V_{2,t+1} \sigma \varepsilon_{t+1} + V_{3,t+1}\right]$$

where:

$$\begin{aligned} k_{t+1} &= e^{z_t} k_t^{\theta} + (1-\delta)\, k_t - c\,(k_t, z_t; \chi) \\ z_t &= \lambda z_{t-1} + \chi \sigma \varepsilon_t,\ \varepsilon_t \sim \mathcal{N}(0,1) \end{aligned}$$

- We take derivatives of each of the four equations w.t.r. k_t, z_t, and χ.
- We take advantage of the equality of cross derivatives.
- The envelope theorem does not hold anymore (we are taking derivatives of the derivatives of the value function).

First Equation I

We have:

$$(1-\beta)\, c\left(k_t, z_t; \chi\right)^{-\gamma} - \beta \mathbb{E}_t V_{1,t+1} = 0$$

- Derivative with respect to k_t:

$$\begin{aligned} &-(1-\beta)\,\gamma c\left(k_t, z_t; \chi\right)^{-\gamma-1} c_{1,t} \\ -\beta \mathbb{E}_t &\left[V_{11,t+1} \left(e^{z_t} \theta k_t^{\theta-1} + 1 - \delta - c_{1,t} \right) \right] = 0 \end{aligned}$$

In steady state:

$$\left(\beta V_{11,ss} - (1-\beta)\,\gamma c_{ss}^{-\gamma-1} \right) c_{1,ss} = \beta \left[V_{11,ss} \left(\theta k_{ss}^{\theta-1} + 1 - \delta \right) \right]$$

or

$$c_{1,ss} = \frac{V_{11,ss}}{\beta V_{11,ss} - (1-\beta)\,\gamma c_{ss}^{-\gamma-1}}$$

where we have used that $1 = \beta \left(\theta k_{ss}^{\theta-1} + 1 - \delta \right)$.

First Equation II

- Derivative with respect to z_t:

$$-\left(1-\beta\right)\gamma c\left(k_t, z_t; \chi\right)^{-\gamma-1} c_{2,t}$$
$$-\beta \mathbb{E}_t \left(V_{11,t+1} \left(e^{z_t} k_t^{\theta} - c_{2,t} \right) + V_{12,t+1} \lambda \right) = 0$$

In steady state:

$$\left(\beta V_{11,ss} - \left(1-\beta\right)\gamma c_{ss}^{-\gamma-1}\right) c_{2,ss} = \beta \left(V_{11,ss} k_t^{\theta} + V_{12,ss} \lambda \right)$$

or

$$c_{2,ss} = \frac{\beta}{\beta V_{11,ss} - \left(1-\beta\right)\gamma c_{ss}^{-\gamma-1}} \left(V_{11,ss} k_{ss}^{\theta} + V_{12,ss} \lambda \right)$$

First Equation III

- Derivative with respect to χ:

$$-\left(1-\beta\right)\gamma c\left(k_t, z_t; \chi\right)^{-\gamma-1} c_{3,t}$$
$$-\beta \mathbb{E}_t\left(-V_{11,t+1} c_{3,t} + V_{12,t+1}\sigma\varepsilon_{t+1} + V_{13,t+1}\right) = 0$$

In steady state:

$$\left(\beta V_{11,ss} - \left(1-\beta\right)\gamma c_{ss}^{-\gamma-1}\right) c_{3,ss} = \beta V_{13,ss}$$

or

$$c_{3,ss} = \frac{\beta}{\left(\beta V_{11,ss} - \left(1-\beta\right)\gamma c_{ss}^{-\gamma-1}\right)} V_{13,ss}$$

Second Equation I

We have:

$$V_{1,t} = \beta \mathbb{E}_t V_{1,t+1} \left(\theta e^{z_t} k_t^{\theta-1} + 1 - \delta \right)$$

- Derivative with respect to k_t:

$$V_{11,t} = \beta \mathbb{E}_t \left[\begin{array}{c} V_{11,t+1} \left(\theta e^{z_t} k_t^{\theta-1} + 1 - \delta - c_{1,t} \right) \left(\theta e^{z_t} k_t^{\theta-1} + 1 - \delta \right) \\ + V_{1,t+1} \theta \left(\theta - 1 \right) e^{z_t} k_t^{\theta-2} \end{array} \right]$$

In steady state:

$$V_{11,ss} = \left[V_{11,ss} \left(\frac{1}{\beta} - c_{1,ss} \right) + \beta V_{1,ss} \theta \left(\theta - 1 \right) k_{ss}^{\theta-2} \right]$$

or

$$V_{11,ss} = \frac{\beta}{1 - \frac{1}{\beta} + c_{1,ss}} V_{1,ss} \theta \left(\theta - 1 \right) k_{ss}^{\theta-2}$$

Second Equation II

- Derivative with respect to z_t:

$$V_{12,t} = \beta \mathbb{E}_t \left[\begin{array}{c} V_{11,t+1} \left(e^{z_t} k_t^{\theta} - c_{2,t} \right) \left(\theta e^{z_t} k_t^{\theta-1} + 1 - \delta \right) \\ + V_{12,t+1} \lambda \left(\theta e^{z_t} k_t^{\theta-1} + 1 - \delta \right) + V_{1,t+1} \theta e^{z_t} k_t^{\theta-1} \end{array} \right]$$

In steady state:

$$V_{12,ss} = V_{11,ss} \left(k_{ss}^{\theta} - c_{2,ss} \right) + V_{12,ss} \lambda + \beta V_{1,ss} \theta k_t^{\theta-1}$$

or

$$V_{12,ss} = \frac{1}{1-\lambda} \left[V_{11,ss} \left(k_{ss}^{\theta} - c_{2,ss} \right) + \beta V_{1,ss} \theta k_{ss}^{\theta-1} \right]$$

Second Equation III

- Derivative with respect to χ:

$$V_{13,t} = \beta \mathbb{E}_t \left[-V_{11,t+1} c_{3,t} + V_{12,t+1} \sigma \varepsilon_{t+1} + V_{13,t+1} \right]$$

In steady state,

$$\begin{aligned} V_{13,ss} &= \beta \left[-V_{11,ss} c_{3,ss} + V_{13,ss} \right] \Rightarrow \\ V_{13,ss} &= \frac{\beta}{\beta - 1} V_{11,ss} c_{3,ss} \end{aligned}$$

but since we know that:

$$c_{3,ss} = \frac{\beta}{\left(\beta V_{11,ss} - (1-\beta) \gamma c_{ss}^{-\gamma-1} \right)} V_{13,ss}$$

the two equations can only hold simultaneously if $V_{13,ss} = c_{3,ss} = 0$.

Third Equation I

We have

$$V_{2,t} = \beta \mathbb{E}_t \left[V_{1,t+1} e^{z_t} k_t^\theta + V_{2,t+1} \lambda \right]$$

- Derivative with respect to z_t:

$$V_{22,t} = \beta \mathbb{E}_t \left[\begin{array}{c} V_{11,t+1} \left(e^{z_t} k_t^\theta - c_{2,t} \right) e^{z_t} k_t^\theta + V_{12,t+1} \lambda e^{z_t} k_t^\theta \\ + V_{1,t+1} e^{z_t} k_t^\theta + V_{21,t+1} \lambda \left(e^{z_t} k_t^\theta - c_{2,t} \right) + V_{22,t+1} \lambda^2 \end{array} \right]$$

In steady state:

$$\begin{aligned} V_{22,t} &= \beta \left[\begin{array}{c} V_{11,ss} \left(k_t^\theta - c_{2,ss} \right) k_{ss}^\theta + V_{12,ss} \lambda k_{ss}^\theta + V_{1,ss} k_{ss}^\theta \\ + V_{21,ss} \lambda \left(k_{ss}^\theta - c_{2,ss} \right) + V_{22,ss} \lambda^2 \end{array} \right] \Rightarrow \\ V_{22,ss} &= \frac{\beta}{1 - \beta \lambda^2} \left[\begin{array}{c} V_{11,ss} \left(k_t^\theta - c_{2,ss} \right) k_{ss}^\theta + 2 V_{12,ss} \lambda k_{ss}^\theta \\ + V_{1,ss} k_{ss}^\theta - V_{12,ss} \lambda c_{2,ss} \end{array} \right] \end{aligned}$$

where we have used $V_{12,ss} = V_{21,ss}$.

Third Equation II

- Derivative with respect to χ:

$$V_{23,t} = \beta \mathbb{E}_t \left[\begin{array}{c} -V_{11,t+1} e^{z_t} k_t^{\theta} c_{3,t} + V_{12,t+1} e^{z_t} k_t^{\theta} \sigma \varepsilon_{t+1} + V_{13,t+1} e^{z_t} k_t^{\theta} \\ -V_{21,t+1} \lambda c_{3,t} + V_{22,t+1} \lambda \sigma \varepsilon_{t+1} + V_{23,t+1} \lambda \end{array} \right]$$

In steady state:

$$V_{23,ss} = 0$$

Fourth Equation

We have

$$V_{3,t} = \beta \mathbb{E}_t \left[V_{2,t+1} \sigma \varepsilon_{t+1} + V_{3,t+1} \right].$$

- Derivative with respect to χ:

$$V_{33,t} = \beta \mathbb{E}_t \left[\begin{array}{c} -V_{21,t+1} c_{3,t} \sigma \varepsilon_{t+1} + V_{22,t+1} \sigma^2 \varepsilon_{t+1}^2 + V_{23,t+1} \sigma \varepsilon_{t+1} \\ -V_{31,t+1} c_{3,t} + V_{32,t+1} \sigma \varepsilon_{t+1} + V_{33,t+1} \end{array} \right]$$

In steady state:

$$V_{33,ss} = \frac{\beta}{1-\beta} V_{22,ss}$$

System I

$$c_{1,ss} = \frac{V_{11,ss}}{\beta V_{11,ss} - (1-\beta)\,\gamma c_{ss}^{-\gamma-1}}$$

$$c_{2,ss} = \frac{\beta}{\beta V_{11,ss} - (1-\beta)\,\gamma c_{ss}^{-\gamma-1}} \left(V_{11,ss} k_{ss}^{\theta} + V_{12,ss}\lambda \right)$$

$$V_{11,ss} = \frac{\beta}{1 - \frac{1}{\beta} + c_{1,ss}} V_{1,ss}\theta\,(\theta-1)\,k_{ss}^{\theta-2}$$

$$V_{12,ss} = \frac{1}{1-\lambda} \left[V_{11,ss} \left(k_{ss}^{\theta} - c_{2,ss} \right) + \beta V_{1,ss} \theta k_{ss}^{\theta-1} \right]$$

$$V_{22,ss} = \frac{\beta}{1-\beta\lambda^2} \left[\begin{array}{c} V_{11,ss}\left(k_t^{\theta} - c_{2,ss}\right) k_{ss}^{\theta} + 2 V_{12,ss}\lambda k_{ss}^{\theta} \\ + V_{1,ss} k_{ss}^{\theta} - V_{12,ss}\lambda c_{2,ss} \end{array} \right]$$

$$V_{33,ss} = \frac{\beta}{1-\beta}\sigma^2 V_{22,ss}$$

plus $c_{3,ss} = V_{13,ss} = V_{23,ss} = 0.$

System II

- This is a system of nonlinear equations.
- However, it has a recursive structure.
- By substituting variables that we already know, we can find $V_{11,ss}$.
- Then, using this results and by plugging $c_{2,ss}$, we have a system of two equations, on two unknowns, $V_{12,ss}$ and $V_{22,ss}$.
- Once the system is solved, we can find $c_{1,ss}$, $c_{2,ss}$, and $V_{33,ss}$ directly.

The Welfare Cost of the Business Cycle

- An advantage of performing the perturbation on the value function is that we have evaluation of welfare readily available.

- Note that at the deterministic steady state, we have:

$$V(k_{ss}, 0; \chi) \simeq V_{ss} + \frac{1}{2} V_{33,ss}$$

- Hence $\frac{1}{2} V_{33,ss}$ is a measure of the welfare cost of the business cycle.

- This quantity is not necessarily negative: it may be positive. For example, in an RBC with leisure choice (Cho and Cooley, 2000).

Our Example

- We know that $V_{ss} = \frac{c_{ss}^{1-\gamma}}{1-\gamma}$.
- We can compute the decrease in consumption τ that will make the household indifferent between consuming $(1-\tau)\, c_{ss}$ units per period with certainty or c_t units with uncertainty.
- Thus:

$$\begin{aligned} \frac{c_{ss}^{1-\gamma}}{1-\gamma} + \frac{1}{2} V_{33,ss} &= \frac{\left(c_{ss}\left(1-\tau\right)\right)^{1-\gamma}}{1-\gamma} \Rightarrow \\ \left(\left(1-\tau\right)^{1-\gamma} - 1\right) c_{ss}^{1-\gamma} &= \left(1-\gamma\right) \frac{1}{2} V_{33,ss} \end{aligned}$$

or

$$\tau = 1 - \left[1 + \frac{1-\gamma}{c_{ss}^{1-\gamma}} \frac{1}{2} V_{33,ss}\right]^{\frac{1}{1-\gamma}}$$

A Numerical Example

- We pick standard parameter values by setting

$$\beta = 0.99, \gamma = 2, \delta = 0.0294, \theta = 0.3, \text{and } \lambda = 0.95.$$

- We get:

$$\begin{aligned} V\left(k_t, z_t; 1\right) &\simeq -0.54000 + 0.00295\left(k_t - k_{ss}\right) + 0.11684 z_t \\ &\quad -0.00007\left(k_t - k_{ss}\right)^2 - 0.00985 z_t^2 \\ &\quad -0.97508\sigma^2 - 0.00225\left(k_t - k_{ss}\right) z_t \\ c\left(k_t, z_t; \chi\right) &\simeq 1.85193 + 0.04220\left(k_t - k_{ss}\right) + 0.74318 z_t \end{aligned}$$

- DYNARE produces the same policy function by linearizing the equilibrium conditions of the problem.
- The welfare cost of the business cycle (in consumption terms) is 8.8475e-005, lower than in Lucas (1987) because of the smoothing possibilities allowed by capital.
- Use as an initial guess for VFI.